COMPUTATION OF NONSTATIONARY HEAT EXCHANGE WITH LAMINAR FLUID FLOW IN PIPES OF ANNULAR CROSS-SECTION

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UDC 532.572.2:506.24

An analytic computation method is presented for the temperature field of a viscous incompressible fluid in the case of laminar flow in a ring-like cylindrical channel of arbitrary profile.

The determination of nonstationary temperature field of a viscous incompressible fluid, in the case of its laminar flow being in an annular cylindrical pipe of arbitrary cross-section, can be reduced to the solution of the energy equation [1]

$$\frac{\partial T}{\partial t}+W(r,\ \varphi,\ t)\frac{\partial T}{\partial x}=\Delta T+F(r,\ \varphi,\ x,\ t) \tag{1}$$

under some appropriate boundary and initial conditions. Employing substitutions

$$\xi=\frac{r-\rho_1(\varphi)}{\rho_2(\varphi)-\rho_1(\varphi)},$$

$$z=x,\ \psi=\varphi,\ \tau=t,\ T(r,\ \varphi,\ x,\ t)=\theta(\xi,\ \psi,\ z,\ \tau)$$

the profile of the region is reduced to the following standard form:

$$0\leqslant\xi\leqslant 1,\ 0\leqslant z<\infty,\ 0\leqslant\psi\leqslant 2\pi,$$

and Eq. (1) becomes

$$\frac{\partial\theta}{\partial\tau}+\bar{w}(\xi,\ \psi,\ \tau)\frac{\partial\theta}{\partial z}=F_1\frac{\partial\theta}{\partial\xi}+F_2\frac{\partial^2\theta}{\partial\xi^2}+F_3\frac{\partial^2\theta}{\partial\xi\partial\psi}+F_4\frac{\partial^2\theta}{\partial\psi^2}+\frac{\partial^2\theta}{\partial z^2}+\bar{F}(\xi,\ \psi,\ z,\ \tau). \tag{2}$$

1. Boundary Conditions of the First Kind. The Galerkin method is used to solve Eq. (2). The boundary and initial conditions are as follows:

$$\theta(\xi,\ \psi,\ z,\ 0)=f_0(\xi,\ \psi,\ z),$$

$$\theta(0,\ \psi,\ z,\ \tau)=f_1(\psi,\ z,\ \tau),$$

$$\theta(1,\ \psi,\ z,\ \tau)=f_2(\psi,\ z,\ \tau).$$

An appropriate solution is sought in a series form

$$\theta^N=\sum_{n=1}^{N}a_n^N(\tau)\varphi_n(\xi,\ \psi,\ z)+\xi(f_2-f_1)+f_1,$$

where φ_n is a complete system of orthogonal functions which satisfy the conditions

$$\varphi_n(0,\ \psi,\ z)=\varphi_n(1,\ \psi,\ z)=0.$$

The coefficients $a_n^N(\tau)$ can be found from the system of equations [4]

$$\frac{da_n^N}{d\tau}+C_{nl}(\tau)a_n^l(\tau)+F_n(\tau)=0,\quad n=1,\ 2,\ \ldots,\ N. \tag{3}$$

Lomonosov State University, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 20, No. 6, pp. 1003-1007, June, 1971. Original article submitted June 9, 1970.

The initial values for the above system are found by expanding the initial temperature distribution into a series of orthogonal coordinate functions φ_n.

The finding of a solution of the system (3) of ordinary linear differential equations does not present any difficulties.

2. Boundary Conditions of the 3rd Kind. In this case the boundary conditions are given by

$$\frac{\partial\theta}{\partial n} = \pm \mathrm{Bi}(\theta - \theta_c) \quad \text{for} \quad r \in \rho_1(\varphi),\ \rho_2(\varphi).$$

The solution is constructed as follows:

$$\theta = \theta_1(\xi, \psi, z, \tau) + \theta_2(\xi, \psi, z, \tau), \tag{4}$$

where the functions θ_i, i = 1, 2, satisfying the conditions

$$\frac{\partial\theta_1}{\partial n} = \theta_1 = 0 \quad \text{for} \quad \xi = 1,$$
$$\frac{\partial\theta_2}{\partial n} = \theta_2 = 0 \quad \text{for} \quad \xi = 0, \tag{5}$$

are sought in the form

$$\theta_i^N = \sum_{n=1}^{N} a_{ni}^N(\tau)\varphi_{ni}(\psi, z, \xi), \quad i = 1, 2. \tag{6}$$

By satisfying in the mean boundary conditions (5), a system of 2N algebraic equations is obtained for the $2N^2$ unknown $a_{ni}^N(\tau)$, i = 1, 2, n = 1, 2, 3, . . . , N.

By using these linear equations some coefficients are now expressed in terms of others, new functions are adopted as coordinate functions and the Galerkin algorithm is applied; in this manner a system is obtained of ordinary differential equations for the coefficients of the expansion.

The Cauchy problem thus obtained for linear differential equations can easily be solved.

The construction of coordinate functions is illustrated by considering problems with boundary conditions of the 3rd kind.

Suppose that it is required to determine the temperature field for a laminar steady flow of a viscous incompressible fluid in a pipe of annular cross-section and under symmetrical boundary conditions which are independent of ψ. For simplicity, one sets $\rho_1(\psi) = 1$; $\rho_2(\psi) = 2$. Having substituted the new variables one finds that the boundary conditions are of the form

$$\frac{\partial\theta}{\partial\xi} = \begin{cases} +\mathrm{Bi}\,\theta & \xi = 1, \\ -\mathrm{Bi}\,\theta & \xi = 0. \end{cases} \tag{7}$$

The solution is found in the form of a truncated series

$$\theta = \sum_{n,k=1}^{2} a_{nk}(\tau)\psi_n(\xi)f_k(z) + \sum_{n,k=1}^{2} b_{nk}(\tau)\varphi_n(\xi)f_k(z), \tag{8}$$

$$\psi_1(\xi) = \xi^2, \quad \psi_2(\xi) = \xi^3 - \frac{5}{6}\xi^2,$$

$$\varphi_1(\xi) = (\xi - 1)^2, \quad \varphi_2(\xi) = (\xi - 1)^3 + \frac{5}{6}(\xi - 1)^2,$$

$$f_1(z) = z\exp(-z), \quad f_2(z) = z^2\exp(-2z) - \frac{8}{2z}f_1(z).$$

These functions were obtained by a Schmidt orthogonalization of the polynomials $\varphi_n = (\xi - 1)^{2+n}$ and $\psi_n = \xi^{2+n}$ respectively.

The series (8) with the new coordinate functions satisfies in the mean the conditions (7) and is as follows:

$$\theta = a_{21}\left[\frac{\mathrm{Bi}-8}{6(2-\mathrm{Bi})}\psi_1(\xi)f_1(z)+\psi_2(\xi)f_1(z)\right]+a_{22}\left[\frac{\mathrm{Bi}-8}{6(2-\mathrm{Bi})}\psi_2(\xi)f_2(z)+\psi_2(\xi)f_2(z)\right]$$

$$+b_{21}\left[\frac{8-\mathrm{Bi}}{6(2-\mathrm{Bi})}\varphi_1(\xi)f_1(z)+\varphi_2(\xi)f_1(z)\right]+b_{22}\left[\frac{8-\mathrm{Bi}}{6(2-\mathrm{Bi})}\varphi_1(\xi)f_2(z)+\varphi_2(\xi)f_2(z)\right].$$

To show how this method can be implemented the following boundary-value problem is considered: the energy equation in dimensionless variables is of the form

$$\frac{\partial\theta}{\partial \mathrm{Fo}}+U(\xi)\frac{\partial\theta}{\partial z}=\frac{1}{\xi+1}\cdot\frac{\partial\theta}{\partial\xi}+\frac{\partial^2\theta}{\partial\xi^2}+\frac{1}{\mathrm{Pe}}\cdot\frac{\partial^2\theta}{\partial z^2},$$

$$\xi=\frac{R-1}{R_2-1},\quad R=\frac{r}{r_1},\quad R_2=\frac{r_2}{r_1},\quad \mathrm{Fo}=\frac{at}{r_1^2},\quad z=\frac{1}{\mathrm{Pe}}\cdot\frac{x}{r_1},$$

$$\mathrm{Pe}=\frac{2U_{\mathrm{av}}r_1}{a},\ R_2=2,\ r_1=1,\ \mathrm{Pe}=1,$$

$$\theta=\frac{T-T_0}{T_{\mathrm{st}}-T_0},\ U(\xi)=\left[4-(\xi+1)^2\right]\ln\frac{1}{2}-3\ln\frac{\xi+1}{2}.$$

The initial conditions are as follows $\theta(0, z, \xi) = 0$.

The boundary conditions are as follows:

$$\theta(1,\ z,\ \mathrm{Fo})=1,$$
$$\theta(0,\ z,\ \mathrm{Fo})=1\quad(0\leqslant z,\ \mathrm{Fo}<\infty),$$
$$\theta(\xi,\ 0,\ \mathrm{Fo})=0\quad(0\leqslant\xi\leqslant 1,\ 0\leqslant \mathrm{Fo}<\infty).$$

The solution is sought in the form

$$\theta^2=\sum_{n,k=1}^{2}a_{nk}(\mathrm{Fo})\psi_n(\xi)f_k(z)+H(\mathrm{Fo})H(z), \tag{9}$$

where H(Fo), H(z) are the Heaviside unit functions and

$$\psi_1(\xi)=(1-\xi)\xi,\quad \psi_2(\xi)=(1-\xi)\xi^2-\frac{1}{2}\psi_1(\xi),$$

$$f_1(z)=z\exp(-z),\quad f_2(z)=z^2\exp(-2z)-\frac{8}{27}f_1(z).$$

The functions $\psi_n(\xi)$ and $f_k(z)$ are obtained by the Schmidt orthogonalization of the systems $\psi_n(\xi) = (1-\xi)\xi^n$, $f_k(z) = z^k\exp(-kz)$, n, k = 1, 2, 3... .

In accordance with the decreased algorithm one obtains the following expressions for the coefficients of the series (9):

$$a_{11}=-1{,}74\left[1-\exp(\lambda_1\mathrm{Fo})\right]-1.33\cdot10^{-4}\left[1-\exp(\lambda_2\mathrm{Fo})\right];$$
$$a_{21}=-0.28\left[1-\exp(\lambda_1\mathrm{Fo})\right]+1.15\cdot10^{-2}\left[1-\exp(\lambda_2\mathrm{Fo})\right];$$
$$a_{12}=-35.9\left[\exp(\lambda_3\mathrm{Fo})-1\right]-5.36\cdot10^{-2}\left[\exp(\lambda_4\mathrm{Fo})-1\right];$$
$$a_{22}=-69.6\left[\exp(\lambda_3\mathrm{Fo})-1\right]+11.56\left[\exp(\lambda_4\mathrm{Fo})-1\right];$$
$$\lambda_1=-11.5;\ \lambda_2=-280.5;\ \lambda_3=-136;\ \lambda_4=-814.$$

It is noticed that in [3] systems can be found of orthonormal functions for a large number of regions which one encounters when solving boundary-value problems of various types. In [4] convergence and stability is proved of the approximating system of differential equations for the Galerkin method.

In the case of stationary problems a_{nk} can be considered as indetermined numerical coefficients and can be determined from a system of linear algebraic equations.

Convergence of the Galerkin method for elliptic equations was proved in [2].

The proposed method can be used to solve heat-exchange problems in solid bodies and to compute the velocity field of one-dimensional motion of viscous incompressible fluid; it is known that the latter is reducible to an equation of the heat-conduction type.

NOTATION

$\Gamma(r, \varphi, x, t)$	is the fluid temperature;
$W(r, \varphi, t)$	is the axial component of fluid flow velocity;
ΔT	is the Laplace operator in cylindrical coordinates;
$F(r, \varphi, x, t)$	is the function characterizing the intensity of heat source;
$\rho_1(\varphi)$, $\rho_2(\varphi)$	are the equations of inner and outer boundary contours of the channel;
$\varphi_n(\xi, \psi, z)$	is the system of orthogonal coordinate functions;
Bi	is the Biot number;
Fo	is the Fourier number;
Pe	is the Peclet number;
H	is the Heaviside unit function;
r, φ, x	are the cylindrical coordinates;
τ	is the time;
ξ, ψ, z	are the new independent variables.

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